Differential Geometry I: Curves and surfaces

1.1 Recall that the cross product of two vectors in $\mathbb{R}^3$, expressed with respect to a positively oriented orthonormal basis $\{e_1, e_2, e_3\}$ as

$$x = x_1e_1 + x_2e_2 + x_3e_3, \quad y = y_1e_1 + y_2e_2 + y_3e_3$$

is the vector

$$x \times y \doteq (x_2y_3 - x_3y_2)e_1 + (x_3y_1 - x_1y_3)e_2 + (x_1y_2 - x_2y_1)e_3.$$

Prove that, for any $a, b \in \mathbb{R}^3$, the cross product $a \times b$ is uniquely determined by the following geometric conditions:

(a) $(a \times b) \perp a$ and $(a \times b) \perp b$,

(b) $\|a \times b\| = \text{Area}(P(a, b))$, where $P(a, b)$ is the parallelogram spanned by the vectors a and b,

(c) If a and b are linearly independent, then $\{a, b, a \times b\}$ is a positively oriented basis of $\mathbb{R}^3$.

Solution. Given two vectors $a, b \in \mathbb{R}^3$, let us denote with $a * b \in \mathbb{R}^3$ the vector defined by properties (a)–(c) above. First of all, let us note that properties (a)–(c) define a unique vector:

- If $\{a, b\}$ are linearly dependent, then $\text{Area}(P(a, b)) = 0$, so $\|a * b\| = 0 \Rightarrow a * b = 0$.
- If $\{a, b\}$ are linearly independent, then propertis (a) and (c) fix a unique *direction* for $a * b$, while property (b) fixes the norm $\|a * b\| \neq 0$. Combined, this information determines a unique non-zero vector $a * b$.

Thus, it remains to show that the unique vector $a * b$ defined by the properties (a)–(c) above is equal to $a \times b$. In particular, we simply have to show that $a \times b$ satisfies properties (a)–(c). To this end, let us fix a positively oriented orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$; if $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ with respect to this basis, then

$$a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

We can then easily compute that

- $\langle a, a \times b\rangle = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0$ and, similarly, $\langle b, a \times b\rangle = 0$,
- Assuming that both a, b are non-zero (otherwise, we trivially have $\text{Area}(P(a, b)) = 0$), we can define the angle $\theta \in [0, \pi]$ between a, b by the relation

  $$\cos(\theta) = \frac{\langle a, b\rangle}{\|a\|\|b\|}.$$

  Then (since $\theta \in [0, \pi]$, we have that $\sin(\theta) \geqslant 0$):

  $$\begin{aligned}\text{Area}(P(a, b))^2 =& \|a\|^2\|b\|^2 \sin(\theta)^2 = \|a\|^2\|b\|^2(1 - \cos^2(\theta) = \|a\|^2\|b\|^2 - \langle a, b\rangle^2 \\ =& (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2\end{aligned}$$

$$= \sum_{i,j\in\{1,2,3\}} a_i^2 b_j^2 - \sum_{i\in\{1,2,3\}} a_i^2 b_i^2 - \sum_{i\neq j\in\{1,2,3\}} a_i b_i a_j b_j$$
$$= \sum_{i\neq j\in\{1,2,3\}} a_i^2 b_j^2 - \sum_{i\neq j\in\{1,2,3\}} a_i b_i a_j b_j$$

and

$$\begin{aligned}\|a\times b\|^2 =& (a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2\\ =& a_2^2b_3^2+a_3^2b_2^2-2a_2b_3a_3b_2+a_3^2b_1^2+a_1^2b_3^2-2a_3b_1a_1b_3+a_1^2b_2^2+a_2^2b_1^2-2a_1b_2a_2b_1\\ =& \sum_{i\neq j\in\{1,2,3\}} a_i^2 b_j^2 - \sum_{i\neq j\in\{1,2,3\}} a_i b_i a_j b_j.\end{aligned}$$

Note that the above expressions are equal to each other.

- In the case when the vectors $\{a,b\}$ are linearly independent, in order to show that $\{a,b,a\times b\}$ is a positive oriented basis, it sufficies to check that the 3×3 matrix with columns formed by the coordinates of $\{a,b,a\times b\}$ has positive determinant:

$$\begin{aligned}\det\big([a,b,a\times b]\big) \quad &= \quad \det\begin{pmatrix} a_1 & b_1 & a_2b_3-a_3b_2\\ a_2 & b_2 & a_3b_1-a_1b_3\\ a_3 & b_3 & a_1b_2-a_2b_1\end{pmatrix}\\ &\overset{\substack{\text{Expand in}\\ 3^{rd}\text{ column}}}{=} \quad (a_2b_3-a_3b_2)(a_2b_3-a_3b_2)-(a_3b_1-a_1b_3)(a_1b_3-a_3b_1)+(a_1b_2-a_2b_1)(a_1b_2-a_2b_1)\\ &= \quad (a_2b_3-a_3b_2)^2+(a_1b_3-a_3b_1)^2+(a_1b_2-a_2b_1)^2.\end{aligned}$$

Remark. The definition of $a\times b$ according to properties $(a)--(c)$ is *geometric*, i.e. independent of any choice of coordinate system. In particular, if $\Phi\in SO(3)$ (i.e. Φ is an isometry that preserves the orientation), we have that
$$\Phi(a\times b)=\Phi(a)\times\Phi(b),$$
, since properties $(a)--(c)$ are all preserved under the action of maps in $SO(3)$. When we need to perform explicit computations involving the cross product $a\times b$, it is sometimes advantageous to choose an orthonormal basis $\{e_1,e_2,e_3\}$ that simplifies the expressions at hand (e.g. choosing e_1 to be in the direction of a).

1.2 Let $\|\cdot\|$ be a norm on the real vector space V. Assume that $\|\cdot\|$ satisfies the *parallelogram law*: For any $x,y\in\mathbb{V}$, we have
$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2.$$
Prove that $\|\cdot\|$ is a Euclidean norm, i.e. there exists an inner product $\langle\cdot,\cdot\rangle$ on V such that $\|x\|=\sqrt{\langle x,x\rangle}$ for all $x\in V$.

Solution. In the case when $\|\cdot\|$ is indeed the norm arising from an inner product $\langle\cdot,\cdot\rangle$, we can recover the expression for $\langle\cdot,\cdot\rangle$ from the identity

$$\langle x, y\rangle = \frac{1}{2}\Big(\|x+y\|^2 - \|x\|^2 - \|y\|^2\Big).$$

In this case, given the norm $\|\cdot\|$ in the statement of the exercise the exercise, let us define the map $\langle\cdot,\cdot\rangle_E : V \times V \to \mathbb{R}$ by the above formula, namely

$$\langle x, y\rangle_E = \frac{1}{2}\Big(\|x+y\|^2 - \|x\|^2 - \|y\|^2\Big). \tag{1}$$

We need to show that $\langle\cdot,\cdot\rangle_E$ is indeed an inner product, namely that it is *symmetric, bilinear* and *positive definite*. Once this is established, it will follow immediately from the above definition that

$$\langle x, x\rangle_E = \|x\|^2, \tag{2}$$

which is what the exercise asks for.

The expression (1) is obviously symmetric in x, y, so $\langle x, y\rangle_E$ is symmetric; it is also positive definite in view of (2) (since $\|\cdot\|$ is a norm, the right hand side of (2) vanishes only when $x = 0$). Thus, we merely have to show that $\langle\cdot,\cdot,\rangle_E$ is bilinear.

We will first show that, for any $x, y, z \in V$, we have

$$\langle x, y+z\rangle_E = \langle x, y\rangle_E + \langle x, z\rangle_E \quad \Leftrightarrow \quad \langle x, y+z\rangle_E - \langle x, y\rangle_E - \langle x, z\rangle_E = 0.$$

Using the definition (1), the above is equivalent to

$$\begin{aligned} \|x+y+z\|^2 - \|x\|^2 - \|y+z\|^2 - \Big(\|x+y\|^2 - \|x\|^2 - \|y\|^2\Big) - \Big(\|x+z\|^2 - \|x\|^2 - \|z\|^2\Big) = 0 \\ \Leftrightarrow \|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 - \|x+y\|^2 - \|x+z\|^2 - \|y+z\|^2 = 0. \end{aligned} \tag{3}$$

We will now use our assumption that $\|\cdot\|$ satisfies the parallelogram law, which can be restated as

$$\|a\|^2 + \|b\|^2 = \frac{1}{2}\|a+b\|^2 + \frac{1}{2}\|a-b\|^2 \quad \text{for all } \ a, b \in V.$$

In view of that, we can express

$$\|x+y+z\|^2 + \|x\|^2 = \frac{1}{2}\|2x+y+z\|^2 + \frac{1}{2}\|y+z\|^2,$$

$$\|y\|^2 + \|z\|^2 = \frac{1}{2}\|y+z\|^2 + \frac{1}{2}\|y-z\|^2,$$

$$-\|x+y\|^2 - \|x+z\|^2 = -\frac{1}{2}\|2x+y+z\|^2 - \frac{1}{2}\|y-z\|^2.$$

Adding the above three identities together, we obtain (3). Thus, we have shown that, for all $x, y, z \in \mathbb{V}$,

$$\langle x, y+z\rangle_E = \langle x, y\rangle_E + \langle x, z\rangle_E. \tag{4}$$

In order to complete the proof that $\langle \cdot, \cdot \rangle_E$ is bilinear, we also have to show that, for any $\lambda \in \mathbb{R}$, we have

$$\langle x, \lambda \cdot y \rangle_E = \lambda \langle x, y \rangle_E. \tag{5}$$

Note that, by applying (4) successively for $z = \pm y$, $z = \pm 2y$ etc, we infer that, for any $x, y \in V$ and any $n \in \mathbb{Z}$:

$$\langle x, n \cdot y \rangle_E = n \langle x, y \rangle_E.$$

Applying the above for the vector $y' = ny$, we also infer that, for any $x, y \in V$ and any $n \in \mathbb{Z}^*$:

$$\langle x, \frac{1}{n} y \rangle_E = \frac{1}{n} \langle x, y \rangle_E.$$

Combining the above, we get for any $m, n \in \mathbb{Z}$, $n \neq 0$:

$$\langle x, \frac{m}{n} \cdot y \rangle_E = \frac{m}{n} \langle x, y \rangle_E,$$

i.e. that () is true for any $\lambda \in \mathbb{Q}$. The relation for $\lambda \in \mathbb{R}$ now follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\langle x, y \rangle_E$ is continuous in y (since $\| \cdot \|$, appearing in the Definition (1), is continuous in its argument).

The isoperimetric inequality in the plane. The aim of Exercises $1.3 - -1.9$ is to establish the isoperimetric inequality in the plane. Let us consider a bounded domain $D \subset \mathbb{R}^2$. Its boundary ∂D is the union of one or more curves, and the perimeter of D is defined as the total length of ∂D (which could be infinite). The isoperimetric quotient of D is defined by

$$\mathrm{Isp}(D) = \frac{(\mathrm{Length}(\partial D))^2}{\mathrm{Area}(D)}.$$

The isoperimetric inequality states that for every bounded domain $D \subset \mathbb{R}^2$, we have

$$\mathrm{Isp}(D) \geqslant \mathrm{Isp}(\mathbb{B}^2),$$

where $\mathbb{B}^2 = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$ is the unit disk. Equality holds if and only if D is a disk (of any radius).

1.3 Let $D \subset \mathbb{R}^2$ be a bounded domain.

(a) Prove that the isoperimetric quotient is invariant under similarity transformations (*Recall that a similarity transformation of a Euclidean space is a bijection which preserves the ratios of distances; it can always be expressed as the composition of a homothetic map $x \to \lambda x$ for a $\lambda > 0$ and an isometry*).

(b) Compute the isoperimetric quotient of a square, of an equilateral triangle, and of a disk.

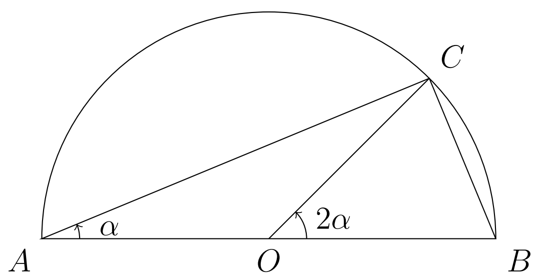
Solution. (a) If two domains D_1 and D_2 of the plane are isometric, then they clearly have the same area and same perimeter. If D_2 is a homothety of D_1, with ratio $\lambda > 0$, then the perimeter of D_2 is equal to λ times the perimeter of D_1 and the area of D_2 is equal to λ^2 times the area of D_1. The two domains therefore have the same isoperimetric quotient.

(b) The isoperimetric quotients of a square, an equilateral triangle, and a disk are respectively 16, $36/\sqrt{3}$, and 4π.

1.4 For this exercise, you will have to recall some basic facts from Euclidean geometry. (a) Let C be a point on the circle with diameter $[A, B]$ (see the figure below). Prove that

$$\widehat{COB} = 2\widehat{CAB}.$$

(b) Deduce the semi-circle theorem of Thales: A triangle ABC is right-angled at C (i.e. $\widehat{ACB} = \frac{\pi}{2}$) if and only if C lies on the circle with diameter $[A, B]$.



Solution. (a) Let us set the following notations:

$$\alpha = \widehat{CAB}, \quad \beta = \widehat{CBA}, \quad \gamma = \widehat{BOC}, \quad \delta = \widehat{AOC}.$$

The triangles OAC and OBC are isosceles (since inscribed in the circle, the sides OA, OB and OC have all length equal to the radius), we therefore also have

$$\alpha = \widehat{OCA}, \quad \beta = \widehat{OCB}, \quad (\alpha + \beta) = \widehat{ACB}.$$

Recall that the sum of the angles of a triangle is equal to π. Applying this fact to the triangles ABC and OBC we see that $2\alpha + 2\beta = \pi$ and $\gamma + 2\beta = \pi$. Consequently $\gamma = \pi - 2\beta = 2\alpha$.

(b) To prove Thales' theorem, we first observe that if the point C is on the circle of diameter $[A, B]$, then, with the previous notations, we have $\widehat{ACB} = (\alpha + \beta) = \pi/2$ since the sum of the angles of the triangle COB is $2\alpha + 2\beta = \pi$. The triangle ABC is therefore a right triangle at C.

To prove the converse, we assume that the point C is not located on the circle of diameter $[A, B]$, and we denote by C' the intersection point of the line OC with the circle. We distinguish two cases: if the point C is outside the circle, then C' is inside the triangle ABC and we have $\widehat{ACB} < \widehat{AC'B} = \pi/2$. If, on the contrary, the point C is inside the circle, then it is the point C that is inside the triangle ABC' and we have $\widehat{ACB} > \widehat{AC'B} = \pi/2$. Finally, $\widehat{ACB} = \pi/2$ if and only if the point C is on the circle of diameter $[A, B]$.

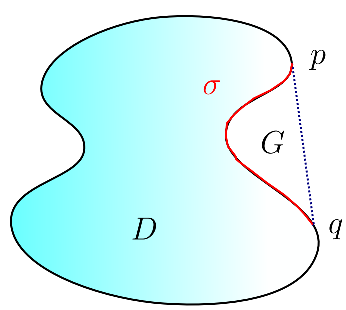
Remark. Because of this construction, we call the *Thales circle of the segment* $[A, B]$ the circle whose diameter is $[A, B]$.

1.5 Prove that among all triangles ABC such that $d(A, C) = x$ and $d(B, C) = y$ with x, y fixed, the one with maximal area is the right triangle with right angle at C.

Solution. The area of triangle ABC is equal to $\frac{1}{2}xy\sin(\gamma)$ where $\gamma = \widehat{ACB}$. This quantity reaches its maximum when $\sin(\gamma) = 1$, therefore when γ is a right angle.

1.6 Recall that a domain $D \subset \mathbb{R}^2$ is called *convex* if, for any two points $A, B \in D$, the line segment $[A, B]$ is contained inside D. Prove that if D is not convex, then D does not minimize the isoperimetric quotient, i.e. you can construct a new domain D' such that $\mathrm{Isp}(D) > \mathrm{Isp}(D')$ (*Hint: Starting from a segment with endpoints on ∂D but not lying entirely in D, show that you can construct a domain D' with Length$(\partial D') \leqslant$ Length(∂D) but Area$(D') >$ Area(D)).*

Solution. If the domain D is not convex, then there exist two points p, q on the boundary ∂D such that the open segment with endpoints p, q lies entirely outside D. Let σ be the curve arc contained in the boundary ∂D bounded by the points $p, q \in \partial D$ such that the open subset G whose boundary is the union of σ and the segment $[p, q]$ is disjoint from D. Then the domain $D' = D \cup G \cup \sigma$ is clearly of greater area than the area of D and its perimeter is smaller because the length of σ is greater than the length of the segment $[p, q]$ (the straight line segment is the shortest path between two points). It is therefore clear that $\mathrm{Isp}(D') < \mathrm{Isp}(D)$.



1.7 Suppose D is an isoperimetrically optimal domain (in particular, D is convex) and let $\Gamma = \partial D$ (since D is convex, Γ is a single connected curve). Suppose that $A, B \in \Gamma$ are two points with the property that they divide Γ into two arcs of equal length. Show that the line segment $[A, B]$ divides D into two regions of equal area.

Solution. We will argue by contradiction. Suppose the chord $[A, B]$ splits D into two regions D_1 and D_2 of unequal areas, for example $\text{Area}(D_1) > \text{Area}(D_2)$. Let D_1' be the region of the plane symmetric to D_1 with respect to the line AB and set $D' = D_1 \cup D_1'$. Then it is clear by construction that D and D' have the same perimeter and

$$\text{Area}(D') = \text{Area}(D_1) + \text{Area}(D_1) > \text{Area}(D_1) + \text{Area}(D_2) = \text{Area}(D).$$

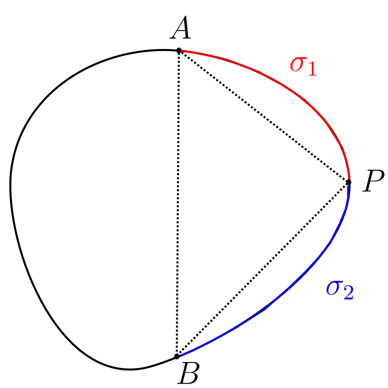
Consequently $\text{Isp}(D') < \text{Isp}(D)$, contradicting the hypothesis that D is an isoperimetric domain.

1.8 Suppose D is an isoperimetrically optimal domain and let the points $A, B \in \partial D$ have the property that they divide ∂D into two arcs of equal length. Show that for every point $P \in \partial D$ distinct from A, B, we have $\widehat{APB} = \frac{\pi}{2}$.

Hint: Assume, for the sake of contradiction, that there exists a $P \in \partial D$ with $\widehat{APB} \neq \frac{\pi}{2}$. Starting from the triangle APB and using Exercise 1.5, show that you can create a domain D' with Length$(\partial D')$ = Length(∂D) but Area(D') > Area(D).

Solution. We will argue again by contradiction. We will prove that if there exists a point $P \in \Gamma = \partial D$ such that $P \neq A, B$ and the triangle ABP is not right-angled at P, then we can construct a domain D' such that $\text{Isp}(D') < \text{Isp}(D)$.

First step: Let σ_1 be the arc of Γ from A to P and σ_2 the arc of Γ from P to B. Then let G_1 be the domain of the plane bounded by the arc σ_1 and the chord $[A, P]$ and G_2 the domain of the plane bounded by the arc σ_2 and the chord $[P, B]$.



Second step: We construct a triangle $A'B'P'$ with a right angle at P' such that $d(A', P') = d(A, P)$ and $d(B', P') = d(B, P)$. By exercise 1.5 we know that the area of $A'B'P'$ is strictly greater than the area of ABP. (Note that $d(A', B') \neq d(A, B)$).

Third step: We add to the triangle $A'B'P'$ a region G_1' isometric to G_1, whose boundary contains the arc A', P' and whose interior is disjoint from the triangle $A'B'P'$. Then we similarly add a region G_2' isometric to G_2, whose boundary contains the arc B', P' and whose interior is disjoint from the triangle $A'B'P'$. We denote by D_1' the obtained domain.

Fourth step: We now denote by D'_2 the domain symmetric to D'_1 with respect to the line $A'B'$, then we set $D' = D'_1 \cup D'_2$. By construction, the perimeter of this domain is double the sum of the lengths of σ_1 and σ_2, so D' and D have the same perimeter. On the other hand, the second step of the construction implies that $\mathrm{Area}(D') > \mathrm{Area}(D')$. We therefore have $\mathrm{Isp}(D') < \mathrm{Isp}(D)$.

1.9 Using the previous exercises, prove the isoperimetric inequality in the plane:

$$Isp(D) \geqslant 4\pi,$$

with equality if and only if D is a disk.

Remark. In proving the above, you can assume that an optimal shape (i.e. a domain D minimizing the value of the isoperimetric ration) exists. Proving the existence of such a minimizer is a harder task and requires the use of more tools from analysis (in particular, a compactness argument analogous to the Arzela–Ascoli theorem).

Solution. Exercises 1.6, 1.8 and 1.4b imply that if D is an isoperimetric domain of the plane, then D is a convex domain whose boundary is a circle.